Date – 04-08-2020

Name of College – S. S. College.
J. Bad

Deptt – Mathematics
Topic – Chord of Curvature
[Curvature, Differential Calculus]
Class – B.Sc (HONS)
Time – 10.15 to 11.00 A.M
11.00 A.M to 11.45 A.M
Date – 04-08-2020
By - Samrendra Kumar

Important Formula :->

(I) Chord of Curvature parallel to x-axis $= 2\rho \sin\psi$

(II) Chord of curvature parallel to y-axis $= 2\rho \cos\psi$

Now :- $\sin\psi$ and $\cos\psi$ will be evaluated from $\tan\psi = \frac{dy}{dx}$

Chord of Curvature through the origin (pole)

(i) Chord of curvature passing through the origin $= 2\rho \sin\phi$

NOTE - $\sin\phi$ will be evaluated from
$$\tan\phi = r\frac{d\theta}{dr}$$

Radius of curvature at the origin –

(I) $\rho = \lim \frac{x^2}{2y}$ [at the origin, x axis being Tangent at the origin]

(II) $\rho = \lim \frac{y^2}{2x}$ [at the origin, y-axis being Tangent at the origin

1.* Problem based on finding the radius of curvature at the origin or by shifting the given point at the origin

1. Find the radius of the curvature of the curve

$$4x^2 - 3xy + y^2 - 3y = 0 \text{ at the origin.}$$

Solution. Here $(0,0)$ satisfies the equation of the curve and $y=0$ is tangent to the curve at the origin.

Therefore, using Newton's method

$$\rho = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2}{2y}$$

Now $4x^2 - 3xy + y^2 - 3y = 0$

$$\frac{4x^2}{2y} - \frac{3xy}{2y} + \frac{y^2}{2y} - \frac{3y}{2y} = 0$$

$$\Rightarrow 4\frac{x^2}{2y} - \frac{3}{2}x + \frac{y}{2} - \frac{3}{2} = 0$$

$$\therefore \lim_{\substack{x\to 0\\ y\to 0}} 4\cdot\frac{x^2}{2y} - \lim_{\substack{x\to 0\\ y\to 0}} \frac{3}{2}x + \lim_{\substack{x\to 0\\ y\to 0}} \frac{y}{2} - \lim_{\substack{x\to 0\\ y\to 0}} \frac{3}{2} = 0$$

$$\Rightarrow 4\rho - 0 - \frac{3}{2} = 0$$

$$\Rightarrow 4\rho = \frac{3}{2}$$

$$\therefore \rho = 3/8$$

(11) Show that radii of curvature at the origin for the curve
$$x^3+y^3 = 3axy \quad \text{are each} = \frac{3a}{2}$$

Solution → These equation of curve is
$$x^3+y^3 = 3axy$$
Clearly it passes through origin

[Since tangents to the curve at the origin are obtained by equating to zero the lowest degree terms in the equation of curve]

Therefore the tangent to the curve at the origin are given by
$$3axy = 0$$
⇒ either $x=0$ or $y=0$

⇒ y- axis or x- axis

When the x- axis is tangent to the curve at origin then
$$\rho = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2}{2y}$$
Now $x^3+y^3 = 3axy$
$$\Rightarrow \frac{x^2}{2y} + \frac{1}{4}xy\cdot\frac{2y}{x^2} = \frac{3a}{2} \qquad \text{Dividing by } 2xy$$
or
$$\lim_{\substack{x\to 0\\ y\to 0}}\left(\frac{x^2}{2y}+\frac{1}{4}xy\frac{2y}{x^2}\right) = \frac{3a}{2}$$
or
$$\lim_{\substack{x\to 0\\ y\to 0}}\frac{x^2}{2y} + \frac{1}{4}\lim_{x\to 0}x\lim_{y\to 0}y\cdot\lim_{\substack{x\to 0\\ y\to 0}}\frac{2y}{x^2} = \frac{3a}{2}$$
or $\rho + \frac{1}{4}\,0\cdot 0\cdot\frac{1}{\rho} = \frac{3a}{2}$

$\Rightarrow \rho = \frac{3a}{2}$

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Case II :→ When the y-axis is Tangent to the curve at the origin, then

$$\ell = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{y^2}{2x}$$

Now $x^3 + y^3 = 3axy$

$$\Rightarrow \frac{x^3}{xy} + \frac{y^3}{xy} = 3a$$

$$\Rightarrow \frac{x^2}{2y} + \frac{y^2}{2x} = \frac{3a}{2}$$

$$\text{or } \frac{1}{4} \frac{\frac{xy}{y^2}}{\frac{y^2}{2x}} + \frac{y^2}{2x} = \frac{3a}{2}$$

$$\text{or } \frac{1}{4} \frac{\left[\lim_{x \to 0} x \lim_{y \to 0} y\right]}{\lim_{\substack{x \to 0 \\ y \to 0}} \frac{y^2}{2x}} + \lim_{\substack{x \to 0 \\ y \to 0}} \frac{y^2}{2x} = \frac{3a}{2}$$

$$\text{or } \frac{1}{4} \cdot \frac{0 \times 0}{\cancel{\ell}} \frac{1}{\ell} + \ell = \frac{3a}{2}$$

$$\text{or } 0 + \ell = \frac{3a}{2}$$

$$\therefore \ell = \frac{3a}{2}$$

Find the radius of curvature of the curve

$$y^2 = x^2 \frac{a+x}{a-x} \text{ at the point } (-a, 0)$$

Solution
Let us shift the origin to the given point $(-a, 0)$ 24-08-2020

for this, write $x-a$ for x and $y+0$ for y in the given equation

$$y^2 = x^2 \frac{a+x}{a-x} \text{ of the Curve}$$

$$\Rightarrow y^2 = \frac{(x-a)^2 \cdot a + x - a}{a-(x-a)}$$

$$\Rightarrow y^2 = \frac{(x-a)^2 \cdot x}{2a-x}$$

To get tangent at the origin, equate the lowest degree terms of (1) to zero

$$\therefore xa^2 = 0 \quad \therefore x = 0$$

this means that y-axis is tangent to the curve (1) at the origin.

$$\therefore \rho = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{y^2}{2x}$$

From (1) $\frac{y^2}{2x} = \frac{1}{2} \frac{(x-a)^2}{2a-x}$

$$\therefore \lim_{\substack{x \to 0 \\ y \to 0}} \frac{y^2}{2x} = \frac{1}{2} \lim_{x \to 0} \frac{(x-a)^2}{2a-x}$$

$$\Rightarrow \rho = \frac{1}{\cancel{2}} \frac{(-a)^2}{\cancel{2}a} = \frac{a}{4}.$$

Problem 4. Show that the radius of curvature of the curve $r = a\sin n\theta$ at the origin is $\frac{na}{2}$.

Solution :→ At the origin,

$$\rho = \lim_{\substack{\theta \to 0 \\ r \to 0}} \frac{r}{2\theta}$$

$$= \lim_{\theta \to 0} \frac{a\sin n\theta}{2\theta}$$

$$= \lim_{\theta \to 0} \frac{a\sin n\theta}{n\theta} \cdot \frac{n}{2}$$

$$= \frac{na}{2}$$